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Some asymptotic expansions of the Eisenstein series

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2 Definition of the Eisenstein series

Let $i = \sqrt{-1}$, $s = \sigma + it \in \mathbb{C}$ and H be the upper half plane. The non-holomorphic Eisenstein series for $SL_2(\mathbb{Z})$ with weight 0 is

$$E(z, s) = y^s \sum_{\{c, d\}} |cz + d|^{-2s}. \quad (1)$$

Here $z = x + iy \in H$, and the summation is taken over $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$, a complete system of representation of $\left\{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \in SL_2(\mathbb{Z})\right\} \setminus SL_2(\mathbb{Z})$. The Fourier expansion is as follows:

$$\begin{aligned} \zeta(2s)E(z, s) &= \zeta(2s)y^s + \sqrt{\pi}\zeta(2s-1)\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}y^{1-s} \\ &\quad + \frac{4\pi^s}{\Gamma(s)}\sqrt{y}\sum_{n=1}^{\infty}n^{s-\frac{1}{2}}\sigma_{1-2s}(n)K_{s-\frac{1}{2}}(2\pi ny)\cos(2\pi nx), \end{aligned} \quad (2)$$

where $K_\nu(\tau)$ is the modified Bessel function and $\sigma_s(n)$ is the sum of s -th powers of positive divisors of n . We call the first two terms of (2) are the constant term of $E(z, s)$.

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3 y -aspect of the Eisenstein series

It is well-known that **the constant term represents the y -aspect of $E(z, s)$ as $y \rightarrow \infty$** . Because the Bessel function in (2) decays exponentially. Therefore there exist positive constants A_1 and A_2 depending only on s such that (except on the poles)

$$|E(z, s)| \leq A_1 y^{\operatorname{Re}(s)} + A_2 y^{1-\operatorname{Re}(s)} \quad (y \rightarrow \infty). \quad (3)$$

The invariance of $E(z, s)$ under the action of $SL_2(\mathbb{Z})$ gives the asymptotic behavior when $y \rightarrow 0$. For every $y > 0$, except on the poles,

$$|E(z, s)| \leq \begin{cases} A_1 (y^{-\operatorname{Re}(s)} + y^{\operatorname{Re}(s)}) & (\operatorname{Re}(s) > \frac{1}{2}) \\ A_2 (y^{-1+\operatorname{Re}(s)} + y^{1-\operatorname{Re}(s)}) & (\operatorname{Re}(s) \leq \frac{1}{2}). \end{cases} \quad (4)$$

The t -aspect of $E(z, s)$ is not simple. **The non-constant terms in (2) are not negligible when $t \rightarrow \infty$.**

Empirically, the behavior of $E(z, s)$ respect to $\operatorname{Im}(s) = t$ is similar to the behavior of $\zeta(s)^2$,

$$E(z, s) \iff \zeta(s)^2.$$

Problem.

Investigate the asymptotic behavior $E(z, s)$ with respect to $\operatorname{Im}(s) = t$.

4 Definition of the Airy Function

The modified Bessel function $K_\nu(\tau)$ ($\nu, \tau \in \mathbb{C}$) is defined by the integral

$$K_\nu(\tau) = \frac{1}{2} \int_0^\infty u^{\nu-1} \exp\left(-\frac{1}{2}\tau\left(u + \frac{1}{u}\right)\right) du,$$

which satisfies the modified Bessel equation:

$$\frac{d^2 w}{d\tau^2} + \frac{1}{\tau} \frac{dw}{d\tau} - \left(1 + \frac{\nu^2}{\tau^2}\right) w = 0.$$

For $\tau \in \mathbb{R}$, the **Airy function** is defined by

$$\operatorname{Ai}(\tau) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{1}{3}u^3 + \tau u\right) du = \frac{1}{\sqrt{3}\pi} \tau^{\frac{1}{2}} K_{\frac{1}{3}}\left(\frac{2}{3}\tau^{\frac{3}{2}}\right),$$

which satisfies the differential equation

$$\frac{d^2 w}{d\tau^2} = \tau w.$$

The representation of $\text{Ai}(\tau)$ for $\tau \in \mathbb{C}$ ($|\arg \tau| < \pi$) is

$$\text{Ai}(\tau) = \frac{\exp\left(-\frac{2}{3}\tau^{\frac{3}{2}}\right)}{2\pi} \int_0^\infty \exp\left(-\tau^{\frac{1}{2}}u\right) \cos\left(\frac{1}{3}u^{\frac{3}{2}}\right) u^{-\frac{1}{2}} du.$$

In which fractional powers take their principal values.

The modified Bessel function $K_{it}(\tau)$ decays exponentially for large τ . The asymptotic expansion of $K_{it}(\tau)$ for the cases $\tau/t \not\sim 1$ or $\tau - t = o(\tau^{1/3})$ are obtainable by using saddle-point method.

However, in the transitional regions, namely τ/t is nearly equal to 1 while $|\tau - t|$ is large, the investigation becomes much more involved. As another approach, the theory of asymptotic solutions of differential equations are employed, where the Airy function plays a fundamental role (cf. [4], 7.4.3, [14]).

5 Main Theorem and Corollaries

Theorem 1 Let $z = x + iy \in H$ and $t > 289$. Assume that $y < t^{\frac{1}{3}-\delta}$ for any positive constant δ , and $N \geq 0$ is any integer satisfying $t - (4 \log t)^{\frac{2}{3}} t^{\frac{1}{3}} \leq 2\pi y N < t$. Define

$$\frac{2}{3}\tau_n = t \log \frac{t + \{t^2 - (2\pi y)^2\}^{\frac{1}{2}}}{2\pi y} - \{t^2 - (2\pi n y)^2\}^{\frac{1}{2}}.$$

Then for every $\varepsilon > 0$,

$$\begin{aligned} E(z, \tfrac{1}{2} + it) &= \frac{4\sqrt{2}\pi^{\frac{1}{2}}y^{\frac{1}{2}}}{\zeta(1+2it)} \sum_{n=1}^N n^{-it} \sigma_{2it}(n) \{t^2 - (2\pi n y)^2\}^{-\frac{1}{4}} \cos(2\pi n x) \tau_n^{\frac{1}{4}} \text{Ai}(-\tau_n^{\frac{2}{3}}) \\ &\quad + y^{\frac{1}{2}+it} + y^{\frac{1}{2}-it} e^{i\theta} + O\left(y^{-\frac{3}{2}}t^{-\frac{1}{3}}(\log t)^{\frac{1}{2}+\varepsilon} + y^{-\frac{1}{2}}(\log t)^{\frac{4}{3}+\varepsilon} \log(t/y)\right). \end{aligned}$$

Here $e^{i\theta} = \pi^{-2it} \zeta(2it) \Gamma(it) / \overline{\zeta(2it) \Gamma(it)}$. The implied O -constant depends at most on ε and δ .

Corollary 1 Suppose $t - t^{\frac{1}{3}+\frac{1}{4}} \leq 2\pi y M \leq t$. For every $\varepsilon > 0$,

$$\begin{aligned} E(z, \tfrac{1}{2} + it) &= \frac{4\sqrt{2}\pi^{\frac{1}{2}}y^{\frac{1}{2}}}{\zeta(1+2it)} \sum_{n=1}^M n^{-it} \sigma_{2it}(n) \{t^2 - (2\pi n y)^2\}^{-\frac{1}{4}} \cos(2\pi n x) \cos(\tfrac{2}{3}\tau_n - \tfrac{\pi}{4}) \\ &\quad + y^{\frac{1}{2}+it} + y^{\frac{1}{2}-it} e^{i\theta} + O\left(y^{-\frac{3}{2}}t^{-\frac{1}{3}}(\log t)^{\frac{1}{2}+\varepsilon} + y^{-\frac{1}{2}}t^{\frac{1}{4}}(\log t)^{\frac{4}{3}+\varepsilon} \log(t/y)\right). \end{aligned}$$

Corollary 2 Let $z = x + iy \in H$. Assume that $c_0 < y < t^{\frac{1}{3}-c_1}$ for some positive constants c_0 and c_1 . Then for every $\varepsilon > 0$,

$$E(z, \tfrac{1}{2} + it) = O\left(y^{-\frac{1}{2}} t^{\frac{1}{2}+\varepsilon}\right) \quad \text{as } t \rightarrow \infty.$$

Remark. Corollary 2 is a convexity bound for the Eisenstein series which includes the y -factor.

6 t -aspect of the Eisenstein series

Known fact 1. A convexity bound is known (see [16] (p. 258)), which is a consequence of the Phragmén-Lindelöf convexity principle;

$$E(z, \tfrac{1}{2} + it) = O_y(t^{\frac{1}{2}+\varepsilon}).$$

Known fact 2. The spectral theory of automorphic forms gives the following estimate:

$$\sum_{0 < t_j < T} |u_j(z)|^2 + \frac{1}{2\pi} \int_0^T |E(z, \tfrac{1}{2} + it)|^2 dt = O(T^2 + Ty).$$

Here $\{u_j\}$ is an orthonormal system of cusp forms for $SL_2(\mathbb{Z})$ with $\Delta u_j = (\frac{1}{4} + t_j^2)u_j$. (See for example [6], (13.1).)

A convexity bound for the Riemann zeta-function is

$$\zeta(\tfrac{1}{2} + it) = O(t^{\frac{1}{4}+\varepsilon}).$$

Further, the sub-convexity bound, the classical result for the Riemann zeta-function due to Hardy-Littlewood is

$$\zeta(\tfrac{1}{2} + it) = O(t^{\frac{1}{6}+\varepsilon}).$$

The mean value theorem of the Riemann zeta-function on the critical line is

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^2 dt = T \log T + (\text{error}).$$

These facts for the Riemann zeta-function support the following conjecture.

The Lindelöf hypothesis For any positive ε ,

$$\zeta(\tfrac{1}{2} + it) = O(t^\varepsilon).$$

(This is true if the Riemann hypothesis is true.)

From the standpoint of the similarity between $E(z, s)$ and $\zeta(s)^2$, we set up the following

Conjecture. Assume $y \geq c_0$ for a positive constant c_0 . For any positive ε ,

$$E(z, \tfrac{1}{2} + it) = O(t^\varepsilon + y^{\frac{1}{2}}) \quad \text{as } t \rightarrow \infty.$$

Here the y -factor comes from the constant term.

7 Jutila's formula

Theorem (Jutila 1984) Let $t \geq 12\pi$, δ be a positive number, $t^\delta \leq N \leq t/12\pi$, and

$$N' = t/2\pi + N/2 - (N^2/4 + Nt/2\pi)^{\frac{1}{2}}.$$

Define

$$f(t, n) = 2t \operatorname{arcsinh} \sqrt{\pi n/2t} + (\pi^2 n^2 + 2\pi n t)^{\frac{1}{2}} + \pi/4.$$

Then,

$$\begin{aligned} |\zeta(\tfrac{1}{2} + it)|^2 &= 2^{\frac{1}{2}} \sum_{n=1}^N (-1)^n d(n) n^{-\frac{1}{2}} \left(\tfrac{1}{4} + \tfrac{t}{2\pi n}\right)^{-\frac{1}{4}} \cos(f(t, n)) \\ &\quad + 2 \sum_{n=1}^{N'} d(n) n^{-\frac{1}{2}} \cos(t \log(t/2\pi n) - t - \tfrac{\pi}{4}) + O\left(N^{\frac{1}{4}} t^{-\frac{1}{4}} (\log t)^2 + \log t\right). \end{aligned}$$

Remark 1. Jutila's formula is a differentiated version of Atkinson's formula. In the proofs of these formulas, Voronoi's summation formula and the saddle-point method are used as the main instruments.

Remark 2. There are some differences between Theorem 1 and the formulas on the square of the Riemann zeta-function. Atkinson type formulas usually have two summations, whereas Theorem 1 and Corollary 1 consist of one summation $\sum_{1 \leq n \leq N}$. This difference is explained by each approximate functional equations;

$$\zeta^2(s) = \sum_{n=1}^N d(n) n^{-s} + \pi^{2s-1} \frac{\Gamma^2(\frac{1}{2} - \frac{s}{2})}{\Gamma^2(\frac{s}{2})} \sum_{n=1}^{N'} d(n) n^{s-1} + O(N^{\frac{1}{2}-\sigma} \log t),$$

where $0 \leq \sigma \leq 1$, $NN' = (t/2\pi)^2$, $N \geq 1$, $N' \geq 1$.

For the case of $E(z, s)$, the Fourier expansion (2) itself may be regarded as one self dual (approximate) functional equation except the constant term. Originally Voronoi's summation formula consists of one summation.

8 Other asymptotic expansions

Define the holomorphic Eisenstein series for $SL_2(\mathbb{Z})$ as

$$E_s(z) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} (m + nz)^{-s}.$$

Theorem 2 (*K. Matsumoto [9]*) Assume $0 < |\arg(z)| < \pi$ and $\operatorname{Re}(s) > -N + 1$ for any positive integer N , then for

$$|z| \geq 1,$$

$$E_s(z) = (1 + e^{\pi i s}) \zeta(s) + O(|z|^{-\operatorname{Re}(s)-N}).$$

For

$$|z| \leq 1,$$

$$\begin{aligned} E_s(z) &= (1 + z^{-s})(1 + e^{\pi i s}) \zeta(s) + (e^{\pi i s} - e^{-\pi i s}) \frac{\zeta(s-1)}{s-1} z^{-1} - \left(1 + \frac{e^{\pi i s} + e^{-\pi i s}}{2}\right) \zeta(s) \\ &\quad + \sum_{1 \leq k \leq N-1, k: \text{odd}} (e^{\pi i s} - e^{-\pi i s}) \binom{-s}{k} \zeta(s+k) \zeta(-k) z^k + O(|z|^N). \end{aligned}$$

Define the non-holomorphic Eisenstein series of weight k attached to $SL_2(\mathbb{Z})$ as

$$E_k(z, s) = \frac{1}{2} \sum_{\substack{c, d = -\infty \\ (c, d) = 1}}^{\infty} (cz + d)^{-k} |cz + d|^{-2s}, \quad (5)$$

and define Ramanujan's Φ -function

$$\Phi_{s_1, s_2}(e(z)) = \sum_{l_1, l_2=1}^{\infty} l_1^{s_1} l_2^{s_2} e(l_1 l_2 z) = \sum_{l=1}^{\infty} \sigma_{s_1-s_2}(l) l^{s_2} e(lz).$$

Theorem 3 (*M. Katsurada [8]*) For any $z \in H$ and any integer $N \geq 0$ the following formula holds in $-N < \operatorname{Re}(s) < 1 + N$ except at $s = 1$.

$$E_0(z, s) = 1 + \frac{\sqrt{\pi} \Gamma(s-1/2) \zeta(2s-1)}{\Gamma(s) \zeta(2s)} y^{1-2s} + \frac{(2\pi)^{2s}}{\Gamma(s) \zeta(2s)} \{S_N(s; z) + R_N(s; z)\}.$$

Here

$$S_N(s; z) = \sum_{n=0}^{N-1} \frac{(-1)^n (s)_n (1-s)_n}{n!} \Phi_{s-n-1, -s-n}^*(e(z)) (4\pi y)^{-s-n}$$

with

$$\Phi_{s_1, s_2}^*(e(z)) = \Phi_{s_1, s_2}(e(z)) + \Phi_{s_1, s_2}(\overline{e(z)}).$$

The remainder term $R_N(s; z)$ is estimated as

$$R_N(s; z) = O\left\{(|t| + 1)^{2N} e^{-2\pi y} y^{-\operatorname{Re}(s)-N}\right\}.$$

Remark. Theorem 3 yields various known results on $E_0(z, s)$, including its functional properties and its asymptotic aspects as $z \rightarrow 0$. Especially the Mellin-Barnes integral transformation shows that the functional equation of $E_0(z, s)$ reduces eventually into the simple property

$$\Phi_{s_1, s_2}(e(z)) = \Phi_{s_2, s_1}(e(z)).$$

Theorem 4 (*M. Katsurada, T. Noda (to appear)*)

$$\begin{aligned} E_k(z, s) &= 1 + (-1)^{k/2} 2\pi \frac{\Gamma(2s+k-1)}{\Gamma(s)\Gamma(s+k)} \frac{\zeta(2s+k-1)}{\zeta(2s+k)} (2y)^{1-2s-k} \\ &\quad + \frac{(-1)^{k/2} (2\pi)^{2s+k}}{\zeta(2s+k)\Gamma(s+k)} \{S_{N+k/2}(s, 2s+k; z) + R_{N+k/2}(s, 2s+k; z)\} \\ &\quad + \frac{(-1)^{k/2} (2\pi)^{2s+k}}{\zeta(2s+k)\Gamma(s)} \{S_{N-k/2}(s+k, 2s+k; -\bar{z}) + R_{N-k/2}(s+k, 2s+k; -\bar{z})\} \end{aligned}$$

holds in the region $-N-k/2 < \operatorname{Re}(s) < N-k/2+1$ except at the complex zeros of $\zeta(2s+k)$ and at the real poles of $E_k(s; z)$. Here the remainder terms are estimated as

$$R_{N+k/2}(s; 2s+k; z) = O\{(|t|+1)^{2N+k} e^{-2\pi y} y^{-\sigma-N-k/2}\}$$

and

$$R_{N-k/2}(s+k; 2s+k; -\bar{z}) = O\{(|t|+1)^{2N-k} e^{-2\pi y} y^{-\sigma-N-k/2}\}.$$

Remark 1. The asymptotic expansion of $E_k(s; z)$ established by transferring from the derived asymptotic expansion of $E_0(s; z)$ (Theorem 3) to that of $E_k(s; z)$ through successive use of Maass' weight change operators.

Remark 2. Theorem 4 also gives a new alternative proof of the Fourier expansion of $E_k(z, s)$, consequently gives new proofs of various results on $E_k(z, s)$, for example, functional equation, special values, the Kronecker limit formula, the eigenfunction equation for the non-Euclidean Laplacian and so on.

9 Outline of the proof of the main theorem

Balogh [3] gave one uniform asymptotic expansion of the modified Bessel function by using Airy functions. Balogh's result is based on Olver's works. The following proposition (Olver [14], Chap.11, p. 425) is the uniform asymptotic expansion of the modified Bessel function of imaginary order, which is crucial in this report.

Proposition 1 ([3], Olver [14] p.425) For $t \in \mathbb{R}_{>0}$, $m \geq 0$ and $u \in \mathbb{C}$ with $|\arg(u)| < \pi$,

$$\begin{aligned} K_{it}(tu) &= \frac{\pi}{t^{1/3}} \exp\left(-\frac{\pi}{2}t\right) \left(\frac{4\xi}{1-u^2}\right)^{1/4} \left\{ \operatorname{Ai}(-t^{2/3}\xi) \sum_{k=0}^m \frac{A_k(\xi)}{t^{2k}} \right. \\ &\quad \left. + t^{-4/3} \operatorname{Ai}'(-t^{2/3}\xi) \sum_{k=0}^{m-1} \frac{B_k(\xi)}{t^{2k}} + \varepsilon_{2m+1}(t, \xi) \right\}. \end{aligned}$$

The Airy function $\text{Ai}(\tau)$ ($\tau \in \mathbb{R}$) decays rapidly as $\tau \rightarrow \infty$, and decays slowly (with oscillation) as $\tau \rightarrow -\infty$. More precisely, we have following

Proposition 2 (I) For $\tau \in \mathbb{C}$ with $|\arg \tau| < \pi$, we have

$$\text{Ai}(\tau) = \frac{\exp\left(-\frac{2}{3}\tau^{\frac{3}{2}}\right)}{2\pi^{\frac{1}{2}}\tau^{\frac{1}{4}}} \sum_{l=0}^{n-1} a_l \left(-\frac{2}{3}\tau^{\frac{3}{2}}\right)^{-l} + (\text{error}).$$

(II) For $\tau \in \mathbb{R}_{>0}$, we have

$$\text{Ai}(-\tau) = \frac{1}{\pi^{\frac{1}{2}}\tau^{\frac{1}{4}}} \sum_{l=0}^{n-1} a_l \cos\left(\frac{2}{3}\tau^{\frac{3}{2}} - \frac{1}{4}\pi - \frac{\pi}{2}l\right) \left(\frac{2}{3}\tau^{\frac{3}{2}}\right)^{-l} + (\text{error}).$$

In (I) and (II), fractional powers of τ take their principal values.

On the critical line $s = \frac{1}{2} + it$, we divide the summation of (2) into five segments:

$$\zeta(1+2it)E(z, \frac{1}{2} + it) = S_0(z, t) + S_1(z, t) + S_2(z, t) + S_3(z, t) + S_\infty(z, t).$$

Here

$$S_0(z, t) = \zeta(1+2it)y^{\frac{1}{2}+it} + \sqrt{\pi}\zeta(2it)\frac{\Gamma(it)}{\Gamma(\frac{1}{2}+it)}y^{\frac{1}{2}-it}.$$

For $j = 1, 2, 3$,

$$S_j(z, t) = 4\pi^{\frac{1}{2}+it}\Gamma(\frac{1}{2}+it)^{-1}\sqrt{y} \sum_{N_{j-1} \leq n < N_j} n^{it} \sigma_{-2it}(n) K_{it}(2\pi ny) \cos(2\pi nx),$$

and

$$S_\infty(z, t) = 4\pi^{\frac{1}{2}+it}\Gamma(\frac{1}{2}+it)^{-1}\sqrt{y} \sum_{n=N_3}^{\infty} n^{it} \sigma_{-2it}(n) K_{it}(2\pi ny) \cos(2\pi nx).$$

Applying the estimations Proposition 1, Proposition 2 and

$$\zeta(1+it)^{-1} = O((\log t)^{\frac{2}{3}}(\log \log t)^{\frac{1}{3}}) \quad (t \geq 2)$$

to $S_j(z, t)$, we obtain the the proof of Theorem 1.

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